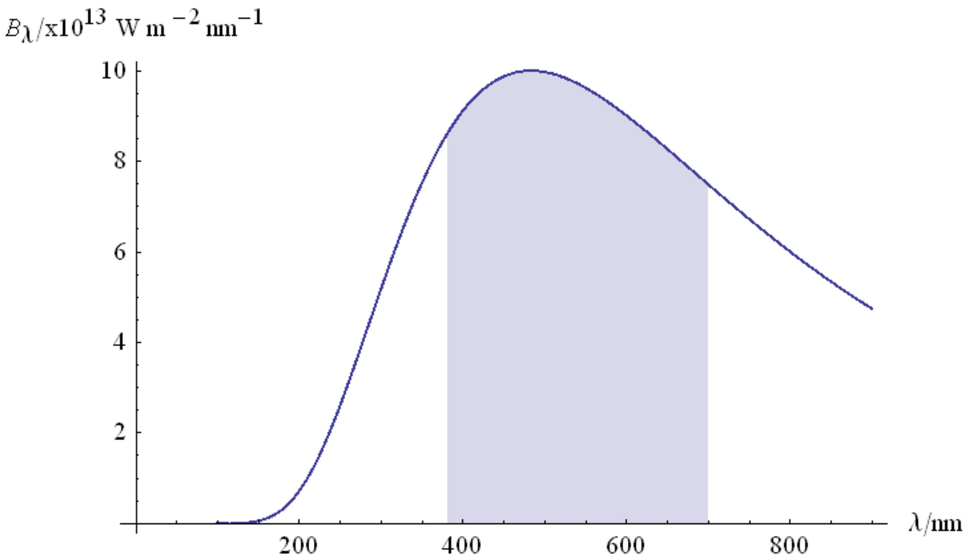
Teacher notes
Topic B

The black body radiation law puzzle – inspired by a question of Edouard Reny on the FB group.

Planck experimentally discovered that the intensity radiated per unit wavelength by a black body at surface temperature T is given by

$$B_\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}$$

B_λ is called spectral intensity or spectral radiance. The intensity (in W m^{-2}) radiated in a wavelength interval $d\lambda$ is $B_\lambda d\lambda$. Hence the total intensity radiated by the body is $I=\int_0^\infty B_\lambda\, d\lambda$ and is given by the area under the curve. The curve shown below corresponds to a temperature of 6000 K, roughly the surface temperature of the Sun.



The wavelength corresponding to the peak of the curve is about 5×10^{-7} m.

The shaded area corresponds to the visible part of the solar spectrum.

We could, however, express the Planck law in terms of frequency. We can find this from

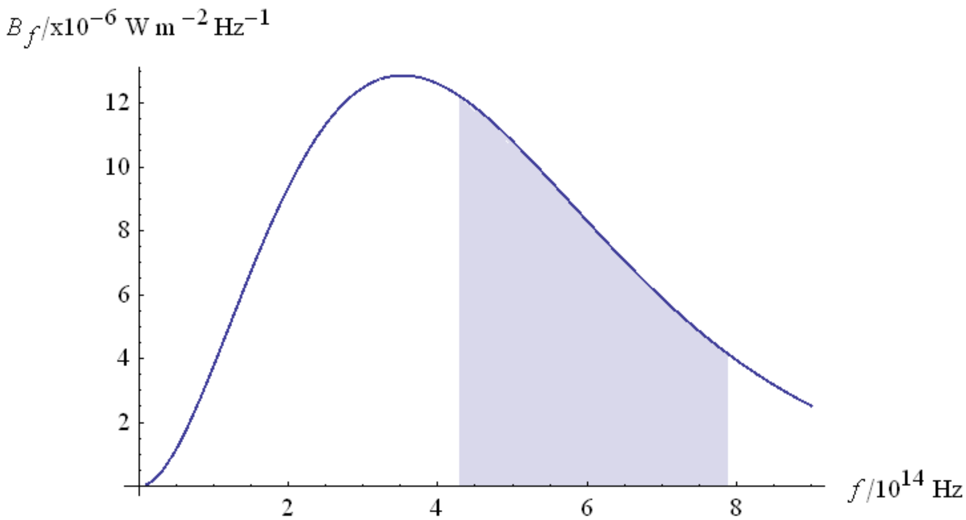
$$B_\lambda d\lambda = B_f df$$

$$B_f = B_\lambda \left|\frac{d\lambda}{df}\right|$$

$$B_f = \frac{2\pi h f^5}{c^3}\frac{1}{e^{\frac{hf}{kT}}-1}\times\frac{c}{f^2}$$ expressed λ in terms of f.

$$B_f = \frac{2\pi h f^3}{c^2}\frac{1}{e^{\frac{hf}{kT}}-1}$$

We can now plot B_f against frequency and we again shade the visible part of the spectrum for T = 6000 K.



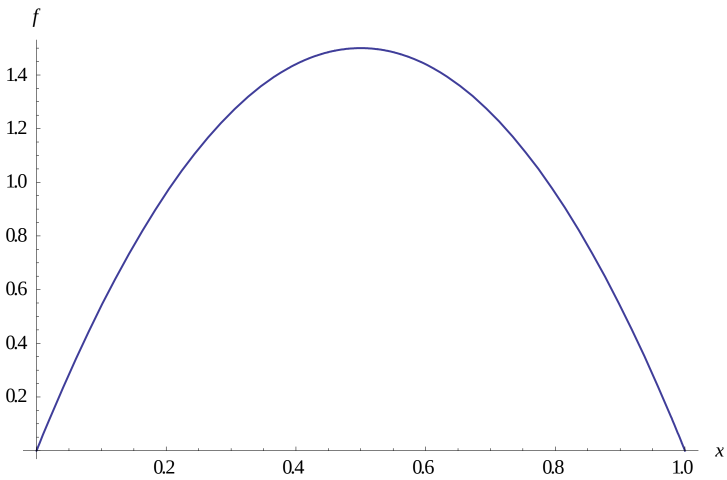
The frequency corresponding to the peak of the curve is about 3.5×10^{14} Hz.

The puzzle is that the product of the peak wavelength and peak frequency is $\lambda_0 f_0 = 5\times10^{-7}\times3.5\times10^{14}\approx1.8\times10^{8}$ m s^{-1} and does not equal the speed of light.

This happens whenever a distribution $f(x)$ is expressed in terms of another variable y, with a non-linear relation between x and y as the following simple example shows:

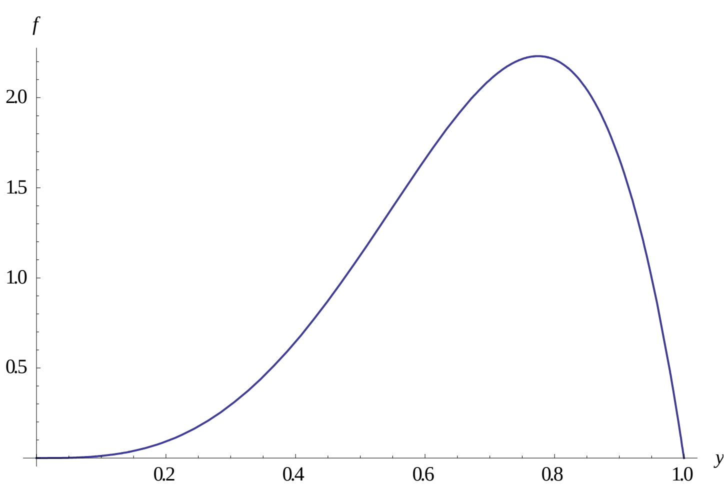
Suppose that we have the distribution $f(x)=6x(1-x)$ for $0\le x\le1$. Notice that $\int_0^1 f(x)=1$ as it should be.

The maximum occurs at $x=\frac{1}{2}$. We change variables to the non-linear $x=y^2$. The graph of $f(x)=6x(1-x)$ is



We must have

$f_x(x)dx = f_y(y)dy \Rightarrow f_y(y) = f_x(x)\dfrac{dx}{dy} = 6y^2(1-y^2)\times 2y = 12y^3(1-y^2)$ for $0 \le y \le 1$. We again have $\int_0^1 f_y(y)\,dy = 1$ as we should. The graph of $f_y(y) = 12y^3(1-y^2)$ is



The maximum occurs at $y^2 = 0.6$ or $y = 0.775$. The two maxima in the two graphs are **not** related by $x = y^2$.

The point is that it is wrong to say that the maximum energy is emitted at the peak wavelength. There is zero energy emitted at **exactly** the peak wavelength or any other wavelength for that matter. The energy is zero because the wavelength interval is zero. It is only meaningful to ask about the energy emitted within an interval of wavelength or frequency. The energy emitted is determined not only by the peak of the curve but also by the width of the interval so looking at the peak alone is meaningless. Clearly, there is less energy emitted in a very narrow interval near the peak than in a very wide interval away from the peak. So, if we ask how much energy is emitted in the visible spectrum, we will get the same answer for both the wavelength and frequency distributions:

$$I = \int_{3.8\times10^{-7}}^{7.0\times10^{-7}} B_\lambda \, d\lambda = 2.9\times10^{7}\ \text{W m}^{-2} = \int_{4.3\times10^{14}}^{7.9\times10^{14}} B_f \, df$$

Similarly, both distributions give the total radiated intensity:

In terms of wavelength

$$I = \int_0^\infty B_\lambda \, d\lambda = \int_0^\infty \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\, d\lambda$$

$$I = \int_\infty^0 \frac{2\pi (kT)^5}{h^4c^3}\frac{x^5}{e^x-1}\frac{hc}{kT}\left(-\frac{1}{x^2}\right)dx$$

$$I = \frac{2\pi (kT)^4}{h^3c^2}\int_0^\infty \frac{x^3}{e^x-1}dx$$

In terms of frequency

$$I = \int_0^\infty B_f \, df = \int_0^\infty \frac{2\pi hf^3}{c^2}\frac{1}{e^{\frac{hf}{kT}}-1}\, df$$

$$I = \int_0^\infty \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^3 \frac{x^3}{e^x-1}\frac{kT}{h}\, dx$$

$$I = \frac{2\pi (kT)^4}{h^3c^2}\int_0^\infty \frac{x^3}{e^x-1}dx$$

The integral is $\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}$ and so $I = \frac{2\pi^5k^4}{15h^3c^2}T^4 = \sigma T^4$ where the Stefan-Boltzmann constant is

$$\sigma = \frac{2\pi^5k^4}{15h^3c^2} = \frac{2\pi^5\times(1.38\times10^{-23})^4}{15\times(6.63\times10^{-34})^3\times(2.998\times10^8)^2} = 5.67\times10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}\ .$$